



FHSST Authors

**The Free High School Science Texts:
Textbooks for High School Students
Studying the Sciences
Mathematics
Grades 10 - 12**

**Version 0
September 17, 2008**

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this a continuously evolving resource!

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Contents

I	Basics	1
1	Introduction to Book	3
1.1	The Language of Mathematics	3
II	Grade 10	5
2	Review of Past Work	7
2.1	Introduction	7
2.2	What is a number?	7
2.3	Sets	7
2.4	Letters and Arithmetic	8
2.5	Addition and Subtraction	9
2.6	Multiplication and Division	9
2.7	Brackets	9
2.8	Negative Numbers	10
2.8.1	What is a negative number?	10
2.8.2	Working with Negative Numbers	11
2.8.3	Living Without the Number Line	12
2.9	Rearranging Equations	13
2.10	Fractions and Decimal Numbers	15
2.11	Scientific Notation	16
2.12	Real Numbers	16
2.12.1	Natural Numbers	17
2.12.2	Integers	17
2.12.3	Rational Numbers	17
2.12.4	Irrational Numbers	19
2.13	Mathematical Symbols	20
2.14	Infinity	20
2.15	End of Chapter Exercises	21
3	Rational Numbers - Grade 10	23
3.1	Introduction	23
3.2	The Big Picture of Numbers	23
3.3	Definition	23

3.4	Forms of Rational Numbers	24
3.5	Converting Terminating Decimals into Rational Numbers	25
3.6	Converting Repeating Decimals into Rational Numbers	25
3.7	Summary	26
3.8	End of Chapter Exercises	27
4	Exponentials - Grade 10	29
4.1	Introduction	29
4.2	Definition	29
4.3	Laws of Exponents	30
4.3.1	Exponential Law 1: $a^0 = 1$	30
4.3.2	Exponential Law 2: $a^m \times a^n = a^{m+n}$	30
4.3.3	Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$	31
4.3.4	Exponential Law 4: $a^m \div a^n = a^{m-n}$	32
4.3.5	Exponential Law 5: $(ab)^n = a^n b^n$	32
4.3.6	Exponential Law 6: $(a^m)^n = a^{mn}$	33
4.4	End of Chapter Exercises	34
5	Estimating Surds - Grade 10	37
5.1	Introduction	37
5.2	Drawing Surds on the Number Line (Optional)	38
5.3	End of Chapter Exercises	39
6	Irrational Numbers and Rounding Off - Grade 10	41
6.1	Introduction	41
6.2	Irrational Numbers	41
6.3	Rounding Off	42
6.4	End of Chapter Exercises	43
7	Number Patterns - Grade 10	45
7.1	Common Number Patterns	45
7.1.1	Special Sequences	46
7.2	Make your own Number Patterns	46
7.3	Notation	47
7.3.1	Patterns and Conjecture	49
7.4	Exercises	50
8	Finance - Grade 10	53
8.1	Introduction	53
8.2	Foreign Exchange Rates	53
8.2.1	How much is R1 really worth?	53
8.2.2	Cross Currency Exchange Rates	56
8.2.3	Enrichment: Fluctuating exchange rates	57
8.3	Being Interested in Interest	58

8.4	Simple Interest	59
8.4.1	Other Applications of the Simple Interest Formula	61
8.5	Compound Interest	63
8.5.1	Fractions add up to the Whole	65
8.5.2	The Power of Compound Interest	65
8.5.3	Other Applications of Compound Growth	67
8.6	Summary	68
8.6.1	Definitions	68
8.6.2	Equations	68
8.7	End of Chapter Exercises	69
9	Products and Factors - Grade 10	71
9.1	Introduction	71
9.2	Recap of Earlier Work	71
9.2.1	Parts of an Expression	71
9.2.2	Product of Two Binomials	71
9.2.3	Factorisation	72
9.3	More Products	74
9.4	Factorising a Quadratic	76
9.5	Factorisation by Grouping	79
9.6	Simplification of Fractions	80
9.7	End of Chapter Exercises	82
10	Equations and Inequalities - Grade 10	83
10.1	Strategy for Solving Equations	83
10.2	Solving Linear Equations	84
10.3	Solving Quadratic Equations	89
10.4	Exponential Equations of the form $ka^{(x+p)} = m$	93
10.4.1	Algebraic Solution	93
10.5	Linear Inequalities	96
10.6	Linear Simultaneous Equations	99
10.6.1	Finding solutions	99
10.6.2	Graphical Solution	99
10.6.3	Solution by Substitution	101
10.7	Mathematical Models	103
10.7.1	Introduction	103
10.7.2	Problem Solving Strategy	104
10.7.3	Application of Mathematical Modelling	104
10.7.4	End of Chapter Exercises	106
10.8	Introduction to Functions and Graphs	107
10.9	Functions and Graphs in the Real-World	107
10.10	Recap	107

10.10.1 Variables and Constants	107
10.10.2 Relations and Functions	108
10.10.3 The Cartesian Plane	108
10.10.4 Drawing Graphs	109
10.10.5 Notation used for Functions	110
10.11 Characteristics of Functions - All Grades	112
10.11.1 Dependent and Independent Variables	112
10.11.2 Domain and Range	113
10.11.3 Intercepts with the Axes	113
10.11.4 Turning Points	114
10.11.5 Asymptotes	114
10.11.6 Lines of Symmetry	114
10.11.7 Intervals on which the Function Increases/Decreases	114
10.11.8 Discrete or Continuous Nature of the Graph	114
10.12 Graphs of Functions	116
10.12.1 Functions of the form $y = ax + q$	116
10.12.2 Functions of the Form $y = ax^2 + q$	120
10.12.3 Functions of the Form $y = \frac{a}{x} + q$	125
10.12.4 Functions of the Form $y = ab^{(x)} + q$	129
10.13 End of Chapter Exercises	133
11 Average Gradient - Grade 10 Extension	135
11.1 Introduction	135
11.2 Straight-Line Functions	135
11.3 Parabolic Functions	136
11.4 End of Chapter Exercises	138
12 Geometry Basics	139
12.1 Introduction	139
12.2 Points and Lines	139
12.3 Angles	140
12.3.1 Measuring angles	141
12.3.2 Special Angles	141
12.3.3 Special Angle Pairs	143
12.3.4 Parallel Lines intersected by Transversal Lines	143
12.4 Polygons	147
12.4.1 Triangles	147
12.4.2 Quadrilaterals	152
12.4.3 Other polygons	155
12.4.4 Extra	156
12.5 Exercises	157
12.5.1 Challenge Problem	159

13 Geometry - Grade 10	161
13.1 Introduction	161
13.2 Right Prisms and Cylinders	161
13.2.1 Surface Area	162
13.2.2 Volume	164
13.3 Polygons	167
13.3.1 Similarity of Polygons	167
13.4 Co-ordinate Geometry	171
13.4.1 Introduction	171
13.4.2 Distance between Two Points	172
13.4.3 Calculation of the Gradient of a Line	173
13.4.4 Midpoint of a Line	174
13.5 Transformations	177
13.5.1 Translation of a Point	177
13.5.2 Reflection of a Point	179
13.6 End of Chapter Exercises	185
14 Trigonometry - Grade 10	189
14.1 Introduction	189
14.2 Where Trigonometry is Used	190
14.3 Similarity of Triangles	190
14.4 Definition of the Trigonometric Functions	191
14.5 Simple Applications of Trigonometric Functions	195
14.5.1 Height and Depth	195
14.5.2 Maps and Plans	197
14.6 Graphs of Trigonometric Functions	199
14.6.1 Graph of $\sin \theta$	199
14.6.2 Functions of the form $y = a \sin(x) + q$	200
14.6.3 Graph of $\cos \theta$	202
14.6.4 Functions of the form $y = a \cos(x) + q$	202
14.6.5 Comparison of Graphs of $\sin \theta$ and $\cos \theta$	204
14.6.6 Graph of $\tan \theta$	204
14.6.7 Functions of the form $y = a \tan(x) + q$	205
14.7 End of Chapter Exercises	208
15 Statistics - Grade 10	211
15.1 Introduction	211
15.2 Recap of Earlier Work	211
15.2.1 Data and Data Collection	211
15.2.2 Methods of Data Collection	212
15.2.3 Samples and Populations	213
15.3 Example Data Sets	213

15.3.1	Data Set 1: Tossing a Coin	213
15.3.2	Data Set 2: Casting a die	213
15.3.3	Data Set 3: Mass of a Loaf of Bread	214
15.3.4	Data Set 4: Global Temperature	214
15.3.5	Data Set 5: Price of Petrol	215
15.4	Grouping Data	215
15.4.1	Exercises - Grouping Data	216
15.5	Graphical Representation of Data	217
15.5.1	Bar and Compound Bar Graphs	217
15.5.2	Histograms and Frequency Polygons	217
15.5.3	Pie Charts	219
15.5.4	Line and Broken Line Graphs	220
15.5.5	Exercises - Graphical Representation of Data	221
15.6	Summarising Data	222
15.6.1	Measures of Central Tendency	222
15.6.2	Measures of Dispersion	225
15.6.3	Exercises - Summarising Data	228
15.7	Misuse of Statistics	229
15.7.1	Exercises - Misuse of Statistics	230
15.8	Summary of Definitions	232
15.9	Exercises	232
16	Probability - Grade 10	235
16.1	Introduction	235
16.2	Random Experiments	235
16.2.1	Sample Space of a Random Experiment	235
16.3	Probability Models	238
16.3.1	Classical Theory of Probability	239
16.4	Relative Frequency vs. Probability	240
16.5	Project Idea	242
16.6	Probability Identities	242
16.7	Mutually Exclusive Events	243
16.8	Complementary Events	244
16.9	End of Chapter Exercises	246
III	Grade 11	249
17	Exponents - Grade 11	251
17.1	Introduction	251
17.2	Laws of Exponents	251
17.2.1	Exponential Law 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$	251
17.3	Exponentials in the Real-World	253
17.4	End of chapter Exercises	254

18 Surds - Grade 11	255
18.1 Surd Calculations	255
18.1.1 Surd Law 1: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$	255
18.1.2 Surd Law 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	255
18.1.3 Surd Law 3: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$	256
18.1.4 Like and Unlike Surds	256
18.1.5 Simplest Surd form	257
18.1.6 Rationalising Denominators	258
18.2 End of Chapter Exercises	259
19 Error Margins - Grade 11	261
20 Quadratic Sequences - Grade 11	265
20.1 Introduction	265
20.2 What is a <i>quadratic sequence</i> ?	265
20.3 End of chapter Exercises	269
21 Finance - Grade 11	271
21.1 Introduction	271
21.2 Depreciation	271
21.3 Simple Depreciation (it really is simple!)	271
21.4 Compound Depreciation	274
21.5 Present Values or Future Values of an Investment or Loan	276
21.5.1 Now or Later	276
21.6 Finding i	278
21.7 Finding n - Trial and Error	279
21.8 Nominal and Effective Interest Rates	280
21.8.1 The General Formula	281
21.8.2 De-coding the Terminology	282
21.9 Formulae Sheet	284
21.9.1 Definitions	284
21.9.2 Equations	285
21.10 End of Chapter Exercises	285
22 Solving Quadratic Equations - Grade 11	287
22.1 Introduction	287
22.2 Solution by Factorisation	287
22.3 Solution by Completing the Square	290
22.4 Solution by the Quadratic Formula	293
22.5 Finding an equation when you know its roots	296
22.6 End of Chapter Exercises	299

23 Solving Quadratic Inequalities - Grade 11	301
23.1 Introduction	301
23.2 Quadratic Inequalities	301
23.3 End of Chapter Exercises	304
24 Solving Simultaneous Equations - Grade 11	307
24.1 Graphical Solution	307
24.2 Algebraic Solution	309
25 Mathematical Models - Grade 11	313
25.1 Real-World Applications: Mathematical Models	313
25.2 End of Chapter Exercises	317
26 Quadratic Functions and Graphs - Grade 11	321
26.1 Introduction	321
26.2 Functions of the Form $y = a(x + p)^2 + q$	321
26.2.1 Domain and Range	322
26.2.2 Intercepts	323
26.2.3 Turning Points	324
26.2.4 Axes of Symmetry	325
26.2.5 Sketching Graphs of the Form $f(x) = a(x + p)^2 + q$	325
26.2.6 Writing an equation of a shifted parabola	327
26.3 End of Chapter Exercises	327
27 Hyperbolic Functions and Graphs - Grade 11	329
27.1 Introduction	329
27.2 Functions of the Form $y = \frac{a}{x+p} + q$	329
27.2.1 Domain and Range	330
27.2.2 Intercepts	331
27.2.3 Asymptotes	332
27.2.4 Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$	333
27.3 End of Chapter Exercises	333
28 Exponential Functions and Graphs - Grade 11	335
28.1 Introduction	335
28.2 Functions of the Form $y = ab^{(x+p)} + q$	335
28.2.1 Domain and Range	336
28.2.2 Intercepts	337
28.2.3 Asymptotes	338
28.2.4 Sketching Graphs of the Form $f(x) = ab^{(x+p)} + q$	338
28.3 End of Chapter Exercises	339
29 Gradient at a Point - Grade 11	341
29.1 Introduction	341
29.2 Average Gradient	341
29.3 End of Chapter Exercises	344

30 Linear Programming - Grade 11	345
30.1 Introduction	345
30.2 Terminology	345
30.2.1 Decision Variables	345
30.2.2 Objective Function	345
30.2.3 Constraints	346
30.2.4 Feasible Region and Points	346
30.2.5 The Solution	346
30.3 Example of a Problem	347
30.4 Method of Linear Programming	347
30.5 Skills you will need	347
30.5.1 Writing Constraint Equations	347
30.5.2 Writing the Objective Function	348
30.5.3 Solving the Problem	350
30.6 End of Chapter Exercises	352
31 Geometry - Grade 11	357
31.1 Introduction	357
31.2 Right Pyramids, Right Cones and Spheres	357
31.3 Similarity of Polygons	360
31.4 Triangle Geometry	361
31.4.1 Proportion	361
31.5 Co-ordinate Geometry	368
31.5.1 Equation of a Line between Two Points	368
31.5.2 Equation of a Line through One Point and Parallel or Perpendicular to Another Line	371
31.5.3 Inclination of a Line	371
31.6 Transformations	373
31.6.1 Rotation of a Point	373
31.6.2 Enlargement of a Polygon 1	376
32 Trigonometry - Grade 11	381
32.1 History of Trigonometry	381
32.2 Graphs of Trigonometric Functions	381
32.2.1 Functions of the form $y = \sin(k\theta)$	381
32.2.2 Functions of the form $y = \cos(k\theta)$	383
32.2.3 Functions of the form $y = \tan(k\theta)$	384
32.2.4 Functions of the form $y = \sin(\theta + p)$	385
32.2.5 Functions of the form $y = \cos(\theta + p)$	386
32.2.6 Functions of the form $y = \tan(\theta + p)$	387
32.3 Trigonometric Identities	389
32.3.1 Deriving Values of Trigonometric Functions for 30° , 45° and 60°	389
32.3.2 Alternate Definition for $\tan \theta$	391

32.3.3	A Trigonometric Identity	392
32.3.4	Reduction Formula	394
32.4	Solving Trigonometric Equations	399
32.4.1	Graphical Solution	399
32.4.2	Algebraic Solution	401
32.4.3	Solution using CAST diagrams	403
32.4.4	General Solution Using Periodicity	405
32.4.5	Linear Trigonometric Equations	406
32.4.6	Quadratic and Higher Order Trigonometric Equations	406
32.4.7	More Complex Trigonometric Equations	407
32.5	Sine and Cosine Identities	409
32.5.1	The Sine Rule	409
32.5.2	The Cosine Rule	412
32.5.3	The Area Rule	414
32.6	Exercises	416
33	Statistics - Grade 11	419
33.1	Introduction	419
33.2	Standard Deviation and Variance	419
33.2.1	Variance	419
33.2.2	Standard Deviation	421
33.2.3	Interpretation and Application	423
33.2.4	Relationship between Standard Deviation and the Mean	424
33.3	Graphical Representation of Measures of Central Tendency and Dispersion	424
33.3.1	Five Number Summary	424
33.3.2	Box and Whisker Diagrams	425
33.3.3	Cumulative Histograms	426
33.4	Distribution of Data	428
33.4.1	Symmetric and Skewed Data	428
33.4.2	Relationship of the Mean, Median, and Mode	428
33.5	Scatter Plots	429
33.6	Misuse of Statistics	432
33.7	End of Chapter Exercises	435
34	Independent and Dependent Events - Grade 11	437
34.1	Introduction	437
34.2	Definitions	437
34.2.1	Identification of Independent and Dependent Events	438
34.3	End of Chapter Exercises	441
IV	Grade 12	443
35	Logarithms - Grade 12	445
35.1	Definition of Logarithms	445

35.2	Logarithm Bases	446
35.3	Laws of Logarithms	447
35.4	Logarithm Law 1: $\log_a 1 = 0$	447
35.5	Logarithm Law 2: $\log_a(a) = 1$	448
35.6	Logarithm Law 3: $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$	448
35.7	Logarithm Law 4: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	449
35.8	Logarithm Law 5: $\log_a(x^b) = b \log_a(x)$	450
35.9	Logarithm Law 6: $\log_a(\sqrt[b]{x}) = \frac{\log_a(x)}{b}$	450
35.10	Solving simple log equations	452
35.10.1	Exercises	454
35.11	Logarithmic applications in the Real World	454
35.11.1	Exercises	455
35.12	End of Chapter Exercises	455
36	Sequences and Series - Grade 12	457
36.1	Introduction	457
36.2	Arithmetic Sequences	457
36.2.1	General Equation for the n^{th} -term of an Arithmetic Sequence	458
36.3	Geometric Sequences	459
36.3.1	Example - A Flu Epidemic	459
36.3.2	General Equation for the n^{th} -term of a Geometric Sequence	461
36.3.3	Exercises	461
36.4	Recursive Formulae for Sequences	462
36.5	Series	463
36.5.1	Some Basics	463
36.5.2	Sigma Notation	463
36.6	Finite Arithmetic Series	465
36.6.1	General Formula for a Finite Arithmetic Series	466
36.6.2	Exercises	467
36.7	Finite Squared Series	468
36.8	Finite Geometric Series	469
36.8.1	Exercises	470
36.9	Infinite Series	471
36.9.1	Infinite Geometric Series	471
36.9.2	Exercises	472
36.10	End of Chapter Exercises	472
37	Finance - Grade 12	477
37.1	Introduction	477
37.2	Finding the Length of the Investment or Loan	477
37.3	A Series of Payments	478
37.3.1	Sequences and Series	479

37.3.2 Present Values of a series of Payments	479
37.3.3 Future Value of a series of Payments	484
37.3.4 Exercises - Present and Future Values	485
37.4 Investments and Loans	485
37.4.1 Loan Schedules	485
37.4.2 Exercises - Investments and Loans	489
37.4.3 Calculating Capital Outstanding	489
37.5 Formulae Sheet	489
37.5.1 Definitions	490
37.5.2 Equations	490
37.6 End of Chapter Exercises	490
38 Factorising Cubic Polynomials - Grade 12	493
38.1 Introduction	493
38.2 The Factor Theorem	493
38.3 Factorisation of Cubic Polynomials	494
38.4 Exercises - Using Factor Theorem	496
38.5 Solving Cubic Equations	496
38.5.1 Exercises - Solving of Cubic Equations	498
38.6 End of Chapter Exercises	498
39 Functions and Graphs - Grade 12	501
39.1 Introduction	501
39.2 Definition of a Function	501
39.2.1 Exercises	501
39.3 Notation used for Functions	502
39.4 Graphs of Inverse Functions	502
39.4.1 Inverse Function of $y = ax + q$	503
39.4.2 Exercises	504
39.4.3 Inverse Function of $y = ax^2$	504
39.4.4 Exercises	504
39.4.5 Inverse Function of $y = a^x$	506
39.4.6 Exercises	506
39.5 End of Chapter Exercises	507
40 Differential Calculus - Grade 12	509
40.1 Why do I have to learn this stuff?	509
40.2 Limits	510
40.2.1 A Tale of Achilles and the Tortoise	510
40.2.2 Sequences, Series and Functions	511
40.2.3 Limits	512
40.2.4 Average Gradient and Gradient at a Point	516
40.3 Differentiation from First Principles	519

40.4	Rules of Differentiation	521
40.4.1	Summary of Differentiation Rules	522
40.5	Applying Differentiation to Draw Graphs	523
40.5.1	Finding Equations of Tangents to Curves	523
40.5.2	Curve Sketching	524
40.5.3	Local minimum, Local maximum and Point of Inflexion	529
40.6	Using Differential Calculus to Solve Problems	530
40.6.1	Rate of Change problems	534
40.7	End of Chapter Exercises	535
41	Linear Programming - Grade 12	539
41.1	Introduction	539
41.2	Terminology	539
41.2.1	Feasible Region and Points	539
41.3	Linear Programming and the Feasible Region	540
41.4	End of Chapter Exercises	546
42	Geometry - Grade 12	549
42.1	Introduction	549
42.2	Circle Geometry	549
42.2.1	Terminology	549
42.2.2	Axioms	550
42.2.3	Theorems of the Geometry of Circles	550
42.3	Co-ordinate Geometry	566
42.3.1	Equation of a Circle	566
42.3.2	Equation of a Tangent to a Circle at a Point on the Circle	569
42.4	Transformations	571
42.4.1	Rotation of a Point about an angle θ	571
42.4.2	Characteristics of Transformations	573
42.4.3	Characteristics of Transformations	573
42.5	Exercises	574
43	Trigonometry - Grade 12	577
43.1	Compound Angle Identities	577
43.1.1	Derivation of $\sin(\alpha + \beta)$	577
43.1.2	Derivation of $\sin(\alpha - \beta)$	578
43.1.3	Derivation of $\cos(\alpha + \beta)$	578
43.1.4	Derivation of $\cos(\alpha - \beta)$	579
43.1.5	Derivation of $\sin 2\alpha$	579
43.1.6	Derivation of $\cos 2\alpha$	579
43.1.7	Problem-solving Strategy for Identities	580
43.2	Applications of Trigonometric Functions	582
43.2.1	Problems in Two Dimensions	582

43.2.2 Problems in 3 dimensions	584
43.3 Other Geometries	586
43.3.1 Taxicab Geometry	586
43.3.2 Manhattan distance	586
43.3.3 Spherical Geometry	587
43.3.4 Fractal Geometry	588
43.4 End of Chapter Exercises	589
44 Statistics - Grade 12	591
44.1 Introduction	591
44.2 A Normal Distribution	591
44.3 Extracting a Sample Population	593
44.4 Function Fitting and Regression Analysis	594
44.4.1 The Method of Least Squares	596
44.4.2 Using a calculator	597
44.4.3 Correlation coefficients	599
44.5 Exercises	600
45 Combinations and Permutations - Grade 12	603
45.1 Introduction	603
45.2 Counting	603
45.2.1 Making a List	603
45.2.2 Tree Diagrams	604
45.3 Notation	604
45.3.1 The Factorial Notation	604
45.4 The Fundamental Counting Principle	604
45.5 Combinations	605
45.5.1 Counting Combinations	605
45.5.2 Combinatorics and Probability	606
45.6 Permutations	606
45.6.1 Counting Permutations	607
45.7 Applications	608
45.8 Exercises	610
V Exercises	613
46 General Exercises	615
47 Exercises - Not covered in Syllabus	617
A GNU Free Documentation License	619

Chapter 16

Probability - Grade 10

16.1 Introduction

Very little in mathematics is truly self-contained. Many branches of mathematics touch and interact with one another, and the fields of probability and statistics are no different. A basic understanding of probability is vital in grasping basic statistics, and probability is largely abstract without statistics to determine the "real world" probabilities.

Probability theory is concerned with predicting statistical outcomes. A simple example of a statistical outcome is observing a head or tail when tossing a coin. Another simple example of a statistical outcome is obtaining the numbers 1, 2, 3, 4, 5, or 6 when rolling a die. (We say one die, many dice.)

For a fair coin, heads should occur for $\frac{1}{2}$ of the number of tosses and for a fair die, each number should occur for $\frac{1}{6}$ of the number of rolls. Therefore, the probability of observing a head on one toss of a fair coin is $\frac{1}{2}$ and that for obtaining a four on one roll of a fair die is $\frac{1}{6}$.

In earlier grades, the idea has been introduced that different situations have different probabilities of occurring and that for many situations there are a finite number of different possible outcomes. In general, events from daily life can be classified as either:

- certain that they will happen; or
- certain that they will not happen; or
- uncertain.

This chapter builds on earlier work and describes how to calculate the probability associated with different situations, and describes how probability is used to assign a number describing the level of chance or the odds associated with aspects of life. The meanings of statements like: 'The HIV test is 85% reliable.' will also be explained.

16.2 Random Experiments

The term *random experiment* or *statistical experiment* is used to describe any repeatable experiment or situation, with individual experiments having one of a set of outcomes or results. A set of outcomes is known as an *event*. For example, the act of tossing a coin or rolling a die can be considered to be simple random experiments. With the results of either a heads or tails or one of $\{1,2,3,4,5,6\}$ being the outcomes. These experiments are repeatable and yield different outcomes each time.

16.2.1 Sample Space of a Random Experiment

The set of all possible outcomes in a random experiment plays an important role in probability theory and is known as the *sample space*. The letter S is used to indicate the sample space.

Using the terminology of set theory, the elements of S are then the outcomes of the random experiment. For example, when tossing a coin the sample space S is made up of $\{heads, tails\}$.



Worked Example 72: Sample Space

Question: What outcomes make up the sample space S when rolling a die.

Answer

Step 3 : Determine all the possible outcomes

The possible outcomes when rolling a die are: 1, 2, 3, 4, 5 and 6.

Step 4 : Define the sample space, S

For rolling a die, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

A set of outcomes is referred to as an *event*. For example, when rolling a die the outcomes that are an even number (i.e. $\{2, 4, 6\}$) would be referred to as an event. It is clear that outcomes and events are subsets of the sample space, S .

A Venn diagram can be used to show the relationship between the outcomes of a random experiment, the sample space and events associated with the outcomes. The Venn diagram in Figure 16.1 shows the difference between the universal set, a sample space and events and outcomes as subsets of the sample space.

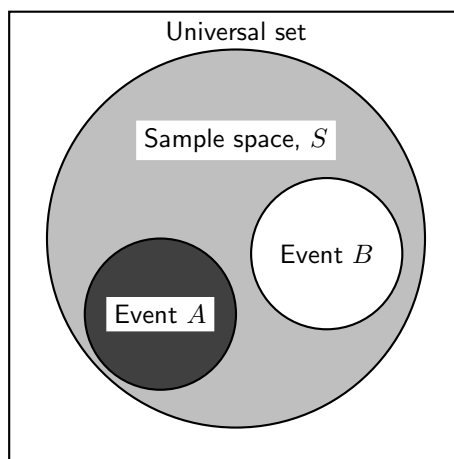


Figure 16.1: Diagram to show difference between the universal set and the sample space. The sample space is made up of all possible outcomes of a statistical experiment and an event is a subset of the sample space.

Venn diagrams can also be used to indicate the union and intersection between events in a sample space (Figure 16.2).



Worked Example 73: Random Experiments

Question: In a box there are pieces of paper with the numbers from 1 to 9 written on them.

$S = \{1; 2; 3; 4; 5; 6; 7; 8; 9\}$

Answer

Step 1 : Consider the events:

- Drawing a prime number; $P = \{2, 3, 5, 7\}$

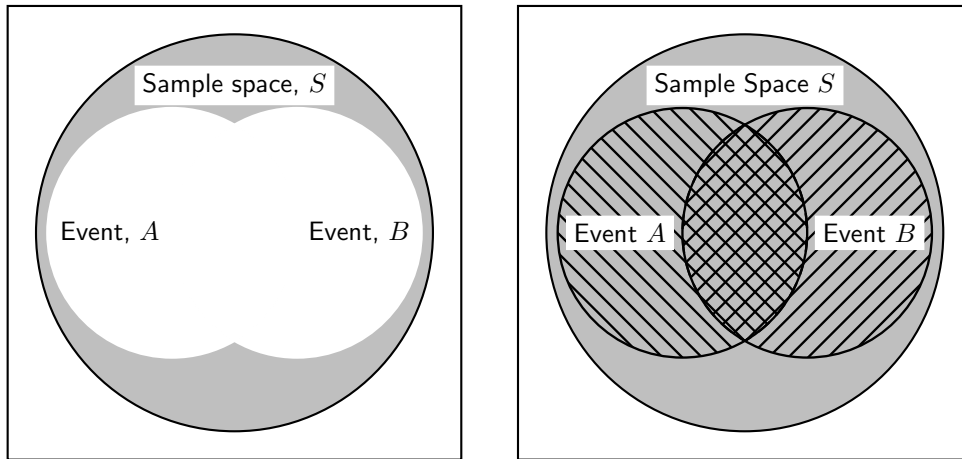
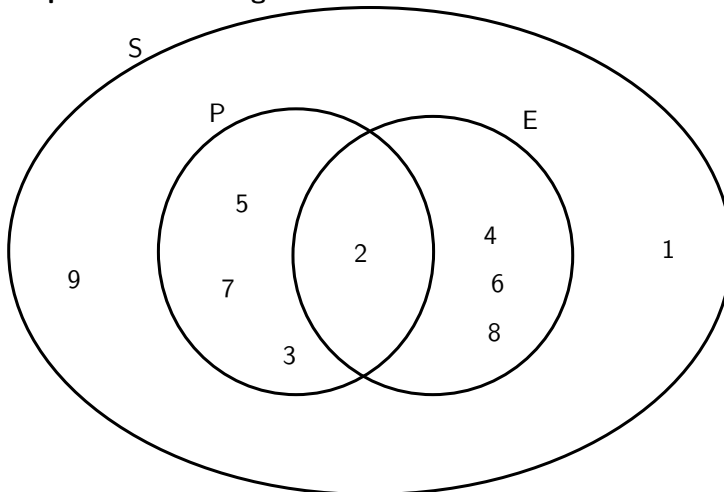


Figure 16.2: Venn diagram to show (left) union of two events, A and B , in the sample space S and (right) intersection of two events A and B , in the sample space S . The crosshatched region indicates the intersection.

- Drawing an even number; $E = \{2, 4, 6, 8\}$

Step 2 : Draw a diagram



Step 3 : Find the union

The *union* of P and E is the set of all elements in P or in E (or in both). P or $E = 2, 3, 4, 5, 6, 7, 8$. P or E is also written $P \cup E$.

Step 4 : Find the intersection

The *intersection* of P and E is the set of all elements in both P and E . P and $E = 2$. P and E is also written as $P \cap E$.

Step 5 : Find the number in each set

We use $n(S)$ to refer to the number of elements in a set S , $n(X)$ for the number of elements in X , etc.

$$\begin{aligned} \therefore n(S) &= 9 \\ n(P) &= 4 \\ n(E) &= 4 \\ n(P \cup E) &= 7 \\ n(P \cap E) &= 2 \end{aligned}$$



Exercise: Random Experiments

1. $S = \{\text{whole numbers from 1 to 16}\}$, $X = \{\text{even numbers from 1 to 16}\}$ and $Y = \{\text{prime numbers from 1 to 16}\}$
 - A Draw a Venn diagram S , X and Y .
 - B Write down $n(S)$, $n(X)$, $n(Y)$, $n(X \cup Y)$, $n(X \cap Y)$.
 2. There are 79 Grade 10 learners at school. All of these take either Maths, Geography or History. The number who take Geography is 41, those who take History is 36, and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take only History.
 - A Draw a Venn diagram to illustrate all this information.
 - B How many learners take Maths and Geography but not History?
 - C How many learners take Geography only?
 - D How many learners take all three subjects?
 3. Pieces of paper labelled with the numbers 1 to 12 are placed in a box and the box is shaken. One piece of paper is taken out and then replaced.
 - A What is the sample space, S ?
 - B Write down the set A , representing the event of taking a piece of paper labelled with a factor 12.
 - C Write down the set B , representing the event of taking a piece of paper labelled with a prime number.
 - D Represent A , B and S by means of a Venn diagram.
 - E Write down
 - i. $n(S)$
 - ii. $n(A)$
 - iii. $n(B)$
 - iv. $n(A \cap B)$
 - v. $n(A \cup B)$
 - F Is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$?
-

16.3 Probability Models

The word probability relates to uncertain events or knowledge, being closely related in meaning to likely, risky, hazardous, and doubtful. Chance, odds, and bet are other words expressing similar ideas.

Probability is connected with uncertainty. In any statistical experiment, the outcomes that occur may be known, but exactly which one might not be known. Mathematically, probability theory formulates incomplete knowledge pertaining to the likelihood of an occurrence. For example, a meteorologist might say there is a 60% chance that it will rain tomorrow. This means that in 6 of every 10 times when the world is in the current state, it will rain.

A probability is a real number between 0 and 1. In everyday speech, probabilities are usually given as a percentage between 0% and 100%. A probability of 100% means that an event is certain, whereas a probability of 0% is often taken to mean the event is impossible. However, there is a distinction between logically impossible and occurring with zero probability; for example, in selecting a number uniformly between 0 and 1, the probability of selecting $1/2$ is 0, but it is not logically impossible. Further, it is certain that whichever number is selected will have had a probability of 0 of being selected.

Another way of referring to probabilities is odds. The odds of an event is defined as the ratio of the probability that the event occurs to the probability that it does not occur. For example, the odds of a coin landing on a given side are $\frac{0.5}{0.5} = 1$, usually written "1 to 1" or "1:1". This means that on average, the coin will land on that side as many times as it will land on the other side.

16.3.1 Classical Theory of Probability

- Equally likely outcomes are outcomes which have an equal chance of happening. For example when a fair coin is tossed, each outcome in the sample space $S = \text{heads, tails}$ is equally likely to occur.
- When all the outcomes are **equally likely** (in any activity), you can calculate the probability of an event happening by using the following definition:
 $P(E) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$
 $P(E) = \frac{n(E)}{n(S)}$
 For example, when you throw a fair dice the possible outcomes are $S = \{1, 2, 3, 4, 5, 6\}$ i.e. the total number of possible outcomes $n(S) = 6$.

Event 1: get a 4

The only possible outcome is a 4, i.e. $E = 4$ i.e. number of favourable outcomes: $n(E) = 1$.

Probability of getting a 4 = $P(4) = \frac{n(E)}{n(S)} = \frac{1}{6}$.

Event 2: get a number greater than 3

Favourable outcomes: $E = \{4, 5, 6\}$

Number of favourable outcomes: $n(E) = 3$

Probability of getting a number more than 3 = $P(\text{more than 3}) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$



Worked Example 74: Classical Probability

Question: Various probabilities relating to a deck of cards

Answer

A standard deck of cards (without jokers) has 52 cards. If we randomly draw a card from the deck, we can think of each card as a possible outcome. Therefore, there are 52 total outcomes. We can now look at various events and calculate their probabilities:

- Out of the 52 cards, there are 13 clubs. Therefore, if the event of interest is drawing a club, there are 13 favourable outcomes, and the probability of this event is $\frac{13}{52} = \frac{1}{4}$.
- There are 4 kings (one of each suit). The probability of drawing a king is $\frac{4}{52} = \frac{1}{13}$.
- What is the probability of drawing a king OR a club? This example is slightly more complicated. We cannot simply add together the number of number of outcomes for each event separately ($4 + 13 = 17$) as this inadvertently counts one of the outcomes twice (the king of clubs). The correct answer is $\frac{16}{52}$.



Exercise: Probability Models

- A bag contains 6 red, 3 blue, 2 green and 1 white balls. A ball is picked at random. What is the probability that it is:
 - red
 - blue or white
 - not green
 - not green or red?
- A card is selected randomly from a pack of 52. What is the probability that it is:

- A the 2 of hearts
 - B a red card
 - C a picture card
 - D an ace
 - E a number less than 4?
3. Even numbers from 2 -100 are written on cards. What is the probability of selecting a multiple of 5, if a card is drawn at random?

16.4 Relative Frequency vs. Probability

There are two approaches to determining the probability associated with any particular event of a random experiment:

1. determining the total number of possible outcomes and calculating the probability of each outcome using the definition of probability
2. performing the experiment and calculating the relative frequency of each outcome

Relative frequency is defined as the number of times an event happens in a statistical experiment divided by the number of trials conducted.

It takes a very large number of trials before the relative frequency of obtaining a head on a toss of a coin approaches the probability of obtaining a head on a toss of a coin. For example, the data in Table 16.1 represent the outcomes of repeating 100 trials of a statistical experiment 100 times, i.e. tossing a coin 100 times.

H	T	T	H	H	T	H	H	H	H
H	H	H	H	T	H	H	T	T	T
T	T	H	T	T	H	T	H	T	H
H	H	T	T	H	T	T	H	T	T
T	H	H	H	T	T	H	T	T	H
H	T	T	T	T	H	T	T	H	H
T	T	H	T	T	H	T	T	H	T
H	T	T	H	T	T	T	T	H	T
T	H	T	T	H	H	H	T	H	T
T	T	T	H	H	T	T	T	H	T

Table 16.1: Results of 100 tosses of a fair coin. H means that the coin landed heads-up and T means that the coin landed tails-up.

The following two worked examples show that the relative frequency of an event is not necessarily equal to the probability of the same event. Relative frequency should therefore be seen as an approximation to probability.

Worked Example 75: Relative Frequency and Probability

Question: Determine the relative frequencies associated with each outcome of the statistical experiment detailed in Table 16.1.

Answer

Step 1 : Identify the different outcomes

There are two unique outcomes: H and T.

Step 2 : Count how many times each outcome occurs.

Outcome	Frequency
H	44
T	56

Step 3 : Determine the total number of trials.

The statistical experiment of tossing the coin was performed 100 times. Therefore, there were 100 trials, in total.

Step 4 : Calculate the relative frequency of each outcome

$$\begin{aligned} \text{Probability of H} &= \frac{\text{frequency of outcome}}{\text{number of trials}} \\ &= \frac{44}{100} \\ &= 0.44 \end{aligned}$$

$$\begin{aligned} \text{Relative Frequency of T} &= \frac{\text{frequency of outcome}}{\text{number of trials}} \\ &= \frac{56}{100} \\ &= 0.56 \end{aligned}$$

The relative frequency of the coin landing heads-up is 0.44 and the relative frequency of the coin landing tails-up is 0.56.

**Worked Example 76: Probability**

Question: Determine the probability associated with an evenly weighted coin landing on either of its faces.

Answer

Step 1 : Identify the different outcomes

There are two unique outcomes: H and T.

Step 2 : Determine the total number of outcomes.

There are two possible outcomes.

Step 3 : Calculate the probability of each outcome

$$\begin{aligned} \text{Relative Frequency of H} &= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} \\ &= \frac{1}{2} \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} \text{Relative Frequency of T} &= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} \\ &= \frac{1}{2} \\ &= 0.5 \end{aligned}$$

The probability of an evenly weighted coin landing on either face is 0.5.

16.5 Project Idea

Perform an experiment to show that as the number of trials increases, the relative frequency approaches the probability of a coin toss. Perform 10, 20, 50, 100, 200 trials of tossing a coin.

16.6 Probability Identities

The following results apply to probabilities, for the sample space S and two events A and B , within S .

$$P(S) = 1 \quad (16.1)$$

$$P(A \cap B) = P(A) \times P(B) \quad (16.2)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (16.3)$$



Worked Example 77: Probability identities

Question: What is the probability of selecting a black or red card from a pack of 52 cards

Answer

$P(S) = n(E)/n(S) = 52/52 = 1$. because all cards are black or red!



Worked Example 78: Probability identities

Question: What is the probability of drawing a club or an ace with one single pick from a pack of 52 cards

Answer

Step 1 : Identify the identity which describes the situation

$$P(\text{club} \cup \text{ace}) = P(\text{club}) + P(\text{ace}) - P(\text{club} \cap \text{ace})$$

Step 2 : Calculate the answer

$$\begin{aligned} &= \frac{1}{4} + \frac{1}{13} - \left(\frac{1}{4} \times \frac{1}{13} \right) \\ &= \frac{1}{4} + \frac{1}{13} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \end{aligned}$$

Notice how we have used $P(C \cup A) = P(C) + P(A) - P(C \cap A)$.



Exercise: Probability Identities

Answer the following questions

1. Rory is target shooting. His probability of hitting the target is 0.7. He fires five shots. What is the probability that:
 - A All five shots miss the center?
 - B At least 3 shots hit the center?

2. An archer is shooting arrows at a bullseye. The probability that an arrow hits the bullseye is 0.4. If she fires three arrows, what is the probability that:
 - A All the arrows hit the bullseye,
 - B only one of the arrows hit the bullseye?

3. A dice with the numbers 1,3,5,7,9,11 on it is rolled. Also a fair coin is tossed.
 - A Draw a sample space diagram to show all outcomes.
 - B What is the probability that:
 - i. A tail is tossed and a 9 rolled?
 - ii. A head is tossed and a 3 rolled?

4. Four children take a test. The probability of each one passing is as follows. Sarah: 0.8, Kosma: 0.5, Heather: 0.6, Wendy: 0.9. What is the probability that:
 - A all four pass?
 - B all four fail?
 - C at least one passes?

5. With a single pick from a pack of 52 cards what is the probability that the card will be an ace or a black card?

16.7 Mutually Exclusive Events

Mutually exclusive events are events, which cannot be true at the same time.

Examples of mutually exclusive events are:

1. A die landing on an even number or landing on an odd number.
2. A student passing or failing an exam
3. A tossed coin landing on heads or landing on tails

This means that if we examine the elements of the sets that make up A and B there will be no elements in common. Therefore, $A \cap B = \emptyset$ (where \emptyset refers to the empty set). Since, $P(A \cap B) = 0$, equation 16.3 becomes:

$$P(A \cup B) = P(A) + P(B)$$

for mutually exclusive events.



Exercise: Mutually Exclusive Events

Answer the following questions

1. A box contains coloured blocks. The number of each colour is given in the following table.

Colour	Purple	Orange	White	Pink
Number of blocks	24	32	41	19

A block is selected randomly. What is the probability that the block will be:

- A purple
 - B purple or white
 - C pink and orange
 - D not orange?
2. A small private school has a class with children of various ages. The table gives the number of pupils of each age in the class.

3 years female	3 years male	4 years female	4 years male	5 years female	5 years male
6	2	5	7	4	6

If a pupil is selected at random what is the probability that the pupil will be:

- A a female
 - B a 4 year old male
 - C aged 3 or 4
 - D aged 3 and 4
 - E not 5
 - F either 3 or female?
3. Fiona has 85 labeled discs, which are numbered from 1 to 85. If a disc is selected at random what is the probability that the disc number:
- A ends with 5
 - B can be multiplied by 3
 - C can be multiplied by 6
 - D is number 65
 - E is not a multiple of 5
 - F is a multiple of 4 or 3
 - G is a multiple of 2 and 6
 - H is number 1?

16.8 Complementary Events

The probability of complementary events refers to the probability associated with events not occurring. For example, if $P(A) = 0.25$, then the probability of A not occurring is the probability associated with all other events in S occurring less the probability of A occurring. This means that

$$P(A') = 1 - P(A)$$

where A' refers to 'not A '. In other words, the probability of 'not A ' is equal to one minus the probability of A .



Worked Example 79: Probability

Question: If you throw two dice, one red and one blue, what is the probability that at least one of them will be a six?

Answer

Step 1 : Work out probability of event 1

To solve that kind of question, work out the probability that there will be no six.

Step 2 : Work out probability of event 2

The probability that the red dice will not be a six is $5/6$, and that the blue one will not be a six is also $5/6$.

Step 3 : Probability of neither

So the probability that neither will be a six is $5/6 \times 5/6 = 25/36$.

Step 4 : Probability of one

So the probability that at least one will be a six is $1 - 25/36 = 11/36$.



Worked Example 80: Probability

Question: A bag contains three red balls, five white balls, two green balls and four blue balls:

1. Calculate the probability that a red ball will be drawn from the bag.
2. Calculate the probability that a ball which is not red will be drawn

Answer

Step 1 : Find event 1

Let R be the event that a red ball is drawn:

- $P(R) = n(R)/n(S) = 3/14$
- R and R' are complementary events

Step 2 : Find the probabilities

$$\therefore P(R') = 1 - P(R) = 1 - 3/14 = 11/14$$

Step 3 : Alternate way to solve it

- Alternately $P(R') = P(B) + P(W) + P(G)$
- $P(R') = 4/14 + 5/14 + 2/14 = 11/14$



Extension: Interpretation of Probability Values

The probability of an event is generally represented as a real number between 0 and 1, inclusive. An impossible event has a probability of exactly 0, and a certain event has a probability of 1, but the converses are not always true: probability 0 events are not always impossible, nor probability 1 events certain. The rather subtle distinction between "certain" and "probability 1" is treated at greater length in the article on "almost surely".

Most probabilities that occur in practice are numbers between 0 and 1, indicating the event's position on the continuum between impossibility and certainty. The closer an event's probability is to 1, the more likely it is to occur.

For example, if two mutually exclusive events are assumed equally probable, such as a flipped or spun coin landing heads-up or tails-up, we can express the probability of each event as "1 in 2", or, equivalently, "50%" or "1/2".

Probabilities are equivalently expressed as odds, which is the ratio of the probability of one event to the probability of all other events. The odds of heads-up, for the tossed/spun coin, are $(1/2)/(1 - 1/2)$, which is equal to $1/1$. This is expressed as "1 to 1 odds" and often written "1:1".

Odds $a:b$ for some event are equivalent to probability $a/(a+b)$. For example, 1:1 odds are equivalent to probability $1/2$, and 3:2 odds are equivalent to probability $3/5$.

16.9 End of Chapter Exercises

- A group of 45 children were asked if they eat Frosties and/or Strawberry Pops. 31 eat both and 6 eat only Frosties. What is the probability that a child chosen at random will eat only Strawberry Pops?
- In a group of 42 pupils, all but 3 had a packet of chips or a Fanta or both. If 23 had a packet of chips and 7 of these also had a Fanta, what is the probability that one pupil chosen at random has:
 - Both chips and Fanta
 - has only Fanta?
- Use a Venn diagram to work out the following probabilities from a die being rolled:
 - A multiple of 5 and an odd number
 - a number that is neither a multiple of 5 nor an odd number
 - a number which is not a multiple of 5, but is odd.
- A packet has yellow and pink sweets. The probability of taking out a pink sweet is $7/12$.
 - What is the probability of taking out a yellow sweet
 - If 44 if the sweets are yellow, how many sweets are pink?
- In a car park with 300 cars, there are 190 Opals. What is the probability that the first car to leave the car park is:
 - an Opal
 - not an Opal
- Tamara has 18 loose socks in a drawer. Eight of these are orange and two are pink. Calculate the probability that the first sock taken out at random is:
 - Orange
 - not orange
 - pink
 - not pink
 - orange or pink
 - not orange or pink
- A plate contains 9 shortbread cookies, 4 ginger biscuits, 11 chocolate chip cookies and 18 Jambos. If a biscuit is selected at random, what is the probability that:
 - it is either a ginger biscuit or a Jambo?
 - it is NOT a shortbread cookie.
- 280 tickets were sold at a raffle. Ingrid bought 15 tickets. What is the probability that Ingrid:
 - Wins the prize
 - Does not win the prize?

9. The children in a nursery school were classified by hair and eye colour. 44 had red hair and not brown eyes, 14 had brown eyes and red hair, 5 had brown eyes but not red hair and 40 did not have brown eyes or red hair.
- A How many children were in the school
 - B What is the probability that a child chosen at random has:
 - i. Brown eyes
 - ii. Red hair
 - C A child with brown eyes is chosen randomly. What is the probability that this child will have red hair
10. A jar has purple, blue and black sweets in it. The probability that a sweet, chosen at random, will be purple is $\frac{1}{7}$ and the probability that it will be black is $\frac{3}{5}$.
- A If I choose a sweet at random what is the probability that it will be:
 - i. purple or blue
 - ii. Black
 - iii. purple
 - B If there are 70 sweets in the jar how many purple ones are there?
 - C $\frac{1}{4}$ of the purple sweets in b) have streaks on them and rest do not. How many purple sweets have streaks?
11. For each of the following, draw a Venn diagram to represent the situation and find an example to illustrate the situation.
- A A sample space in which there are two events that are not mutually exclusive
 - B A sample space in which there are two events that are complementary.
12. Use a Venn diagram to prove that the probability of either event A or B occurring is given by: (A and B are not exclusive)
 $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
13. All the clubs are taken out of a pack of cards. The remaining cards are then shuffled and one card chosen. After being chosen, the card is replaced before the next card is chosen.
- A What is the sample space?
 - B Find a set to represent the event, P, of drawing a picture card.
 - C Find a set for the event, N, of drawing a numbered card.
 - D Represent the above events in a Venn diagram
 - E What description of the sets P and N is suitable? (Hint: Find any elements of P in N and N in P.)
14. Thuli has a bag containing five orange, three purple and seven pink blocks. The bag is shaken and a block is withdrawn. The colour of the block is noted and the block is replaced.
- A What is the sample space for this experiment?
 - B What is the set describing the event of drawing a pink block, P?
 - C Write down a set, O or B, to represent the event of drawing either a orange or a purple block.
 - D Draw a Venn diagram to show the above information.

Appendix A

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